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THE MOTION OF SUBMARINE BOATS IN THE VERTICAL PLANE.

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THE MOTION OF SUBMARINE BOATS IN THE VERTICAL PLANE.

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In order that a submarine boat should be under perfect control in the vertical plane, it should, when going on the surface, be able to dive quickly to any desired depth of immersion. It should be able to keep that depth with certainty and facility within narrow limits, and should be capable of again emerging quickly to the surface at any time.

The boat must therefore possess the two qualities—Stability of Motion and Manœuvring Power. Each of these qualities is examined in the following pages, and also the manner in which they are affected by varying the rudders, the amount and distribution of buoyancy, the shape of hull, &c. But, in order to form a precise opinion on these somewhat complicated questions, it is necessary to treat the subject mathematically.

The mathematical investigation falls into two parts, steady motion and disturbed motion, each of which is treated separately, and the latter in its various forms.

In order to facilitate the mathematical treatment and the subsequent discussion of varying dispositions, it has been found advantageous to start with the following suppositions, which embody the simplest case:—

The boat is perfectly symmetrical about a vertical and a horizontal longitudinal plane, and is propelled by one screw, having its shaft in the intersection of these two planes—the axis of the boat.

There is one rudder, which is placed aft.

The buoyancy exceeds the weight of the boat by a small amount; the surplus buoyancy generally not exceeding one five-hundredth part of the displacement.

A conning tower is placed vertically above the centre of gravity, and the boat is balanced so that, when floating at rest on the surface its axis is horizontal, the top line of its hull just in, or parallel to, the water-line, and only the conning tower, or part of the conning tower, projecting above the surface. When the boat dives, the displacement of that part of the conning tower which projected above water when at rest will constitute the surplus buoyancy. The slight non-symmetry due to the conning tower is at first neglected.

Steady Motion.

We shall first consider the important case of steady motion in a straight line on a certain depth of immersion.*

In order that steady motion may be possible, the centre of gravity should be fixed in position, the weight and buoyancy, as well as the speed of the boat, should be constant, and no external forces should be acting. On account of the symmetry of the boat, we need only consider forces lying in a vertical plane containing the axis.

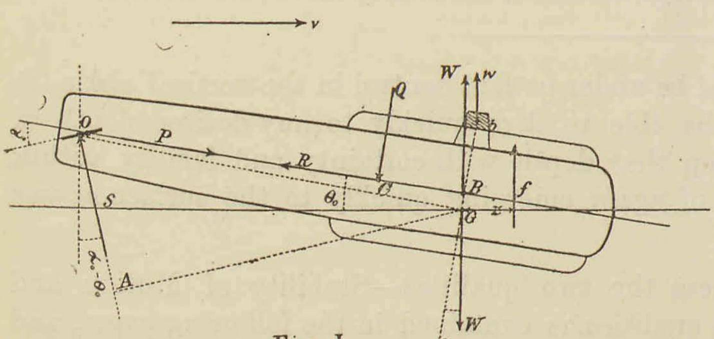


Fig. I.

Let the velocity of the boat, along its horizontal trajectory, be v.

In order to counteract the surplus buoyancy, the boat must be inclined downwards by the head, at a small angle, θ_0, to the horizon. As the boat must, necessarily, possess stability, the rudder must be carried at a certain small angle, a_0, downwards, in order to keep the boat thus inclined.

Both a_0 and θ being small, we shall here, and in what follows, substitute the angles in circular measure for their sines, and put their cosines equal to 1.

The forces which act on the boat, and which, in steady motion, must be in equilibrium, are the following :—

The weight of the boat, W, acting vertically downwards through the centre of gravity, G, situated at a small distance, B G, below the centre of buoyancy.

The buoyancy of the boat, W, exclusive of the surplus buoyancy, acting vertically upwards through the centre of buoyancy, B, which, from symmetry, must be situated in the axis of the boat.

* This part of the problem has been formerly treated in a somewhat different way by a French Naval Architect, M. Leflaive, Ingénieur de la Marine, in "Memorial de Génie Maritime," 1896. Besides the question of steady motion in a horizontal line, M. Leflaive discusses also the case of small oscillations, but does not include any form of resistance, excepting that due to the horizontal translatory motion. His able work has formed a valuable point of issue for the following investigation, and should be studied by all who take an interest in this subject.

The surplus buoyancy, w, acting vertically upwards, through the centre of buoyancy, b, of that part of the conning tower, which projects above water when at rest.

The resistance of the water to the motion of translation. The line of action of this force will cut the axis at some point, C, which, in the following, is referred to as the centre of lateral resistance.

The position of this point will generally vary with the angle of inclination, but we do not know according to what law. It has been taken in the following to remain fixed, at a distance, a, from the centre of buoyancy, a supposition which does not affect the question of steady motion, but which will necessarily make the solution for disturbed motion deviate to a certain extent from reality.

The force of resistance is resolved into two components, R along the axis, and Q, the lateral resistance, at right angles to the same. Q is taken to vary as $\sin \theta$, therefore approximately: $Q = Q_0 \theta$, an assumption which appears justifiable for small values of θ.

The thrust of the propeller, P, acting along the axis.

The pressure of the water on the horizontal rudder, S, supposed acting through the centre of gravity, O, of the rudder area.

The resistance of the rudder, when in its central position, is included in Q. Thus we may put—

$$S = S_0 \alpha_0.$$

The equations of steady motion show that, if the motion is to be stable, we must have—

$$Q_0 a + W \cdot BG > 0.$$

$Q_0 a$, which is denoted by β, measures the moment of the lateral resistance about the centre of gravity, in the same way as $W \cdot BG$ measures the longitudinal stability. $W \cdot BG$ is in the following denoted by ϵ. It is, in a completely submerged boat, also equal to the transverse stability, and must necessarily always be positive.

$\gamma = \beta + \epsilon$ is a measure of the degree of stability of the motion, and is, in the following, supposed > 0.

Disturbed Motion.

A submarine boat may, while submerged, be subject to various disturbances, which will be here discussed in a general way, and which are in the Appendix treated mathematically.

The principal causes of disturbance are:—Faulty use of horizontal rudder; admission of water through leakages; expulsion of foul air and products of combustion;

firing of torpedoes and projectiles; movements of crew; existence of free surfaces of liquid; movements of loose weights, such as fuel; variations in buoyancy caused by varying density of sea-water; grounding and collision; variations in speed.

These disturbing elements can hardly all be entirely avoided, and only experience and training can teach the crew of a submarine boat how to counteract them. We may, however, form an idea of their magnitude, and consider the possibility of avoiding or reducing them, and we may, by means of a mathematical investigation, learn the character of the disturbed motion.

We may, thereby, also be enabled to estimate the relative importance of the various sources of disturbance.

Faulty action of horizontal rudder. If, when in steady motion, the rudder is set at an angle α, differing slightly from α_0, the force of the rudder, S, will vary, both in direction and magnitude. The variation in direction is here disregarded, the angle $(\alpha-\alpha_0)$ being small. The effect of such disturbance may therefore be represented by a vertical force $S(\alpha-\alpha_0)$, acting at a distance p from the centre of gravity.

Admission of water through leakages may, by careful construction of hull and by a strict control over all sea-valves, be reduced to insignificance as long as the boat is not damaged.

Expulsion of foul air need not generally take place during the short period of immersion. Should the necessity arise, however, for drawing on the store of compressed air, and for expelling foul air, the expelled weights will always be small, and the air reservoirs (or chemical stores) used for renewal may be so placed, that no change in the longitudinal balance is caused.

Expulsion of products of combustion, when a motive power is used under water, which, as the various heat-engines, consume both air and fuel. Here the changes in weight would be so serious, that it would become necessary to provide special automatic appliances for always preserving the constancy of the weight and balance.

Electric accumulators are, in this respect, preferable to heat-engines.

Firing of torpedoes from an under-water tube, to which the water has free access before discharge, will cause no sensible change in weight, because the torpedo differs very little in weight from the water, which, after discharge, rushes in and takes its place in the tube, but when a fresh torpedo is afterwards entered into the tube, compensation will be necessary.

Firing of projectiles may cause considerable disturbance, and should be compensated for by the automatic admission of an equal weight of water.

Movements of crew may be avoided by so arranging the service of the various apparatus and machinery, that everybody has to remain immovable at his post.

Free liquid surfaces may exist in various ways :—

Bilge-water may be avoided by carefully pumping dry before diving. The free liquid surfaces which must exist in tubulous boilers, condensers, and accumulators are insignificant. Free water in ballast tanks is a most serious and likely cause of disturbance, and is discussed separately in a later section. By a free use of air-pipes, draining the air from all corners of the tanks, it must be possible to ensure their being completely filled with water.

Shifting of weights when using a heat-engine for under-water propulsion can hardly be avoided. Feed-water, cooling water, compressed air, liquid fuel, or other weights, would be moved about inside the boat, and would have to be compensated for.

Variation in buoyancy may be caused by navigating in water of varying density, as when passing from a river into the sea, or *vice versâ*. In this way an extra force acting through the centre of buoyancy will be caused, which may amount to 1½ per cent. of the displacement. In cases where such disturbance is likely to occur, a special tank should be provided for compensation.

Grounding. In an ordinary vessel the whole energy of motion is absorbed in the act of grounding, partly in lifting the weight of the ship, partly in friction and damage to the bottom, which actions continue until the vessel is brought to a standstill.

But, in the submarine boat the weight is eliminated by the buoyancy, and the shock is due only to the change of momentum at right angles to the bottom of the sea; the velocity parallel to the bottom remaining almost unchanged, if the angle of incidence is not very great.

Thus the friction, and, therefore, the retardation of speed, will generally be small, and the main result will be a vertical, or nearly vertical force, impulsive in character, which will produce a certain velocity, rotatory and translatory, and will then cease to act.

Collision, which is probably far more dangerous to the submarine boat than grounding, will likewise generally produce an impulsive force.

Variations in speed, besides being caused by grounding and collisions, may also be caused by the act of diving, or by variations in the propelling force. The effect will be a corresponding variation in the resistance and in the action of the rudder, *i.e.*, a variation both in the horizontal and vertical forces which act on the boat.

It will be seen that nearly all the causes of disturbance here enumerated may be either avoided, or reduced to small magnitude. Consequently, terms involving second, or higher powers of the deviations, or deflections, may be neglected. The assumption that the deflections are small is justified also by the circumstance that, before a deflection has reached any magnitude, it will be counteracted by the rudder.

It is shown that all the disturbances, excepting the three last-mentioned, may be represented in a general way by a vertical force, f, and a couple, $f z$, with horizontal transverse axis. In the three last-named cases horizontal disturbances also occur, but their effect will be of interest only in so far as they may indirectly produce changes in the disturbing vertical force and in the disturbing couple.

The disturbed motion will cause variations to take place in the resistance of the water. In steady motion, the inclination of the axis to the trajectory of the centre of gravity is always θ_0. But, in disturbed motion, the boat may turn about a transverse axis, and the centre of gravity may move upwards, or downwards, following a curved and possibly undulating path, whereby a variation in the lateral resistance will be caused.

If the inclination to the horizon at any instant is θ, the angle between the axis and the tangent to the trajectory is now approximately $\theta + \frac{1}{v}\frac{d y}{d t}$, assuming the velocity along the trajectory to be always equal to v, an assumption, the justification of which is examined in the Appendix.

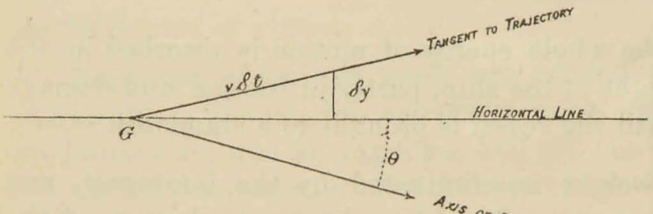


FIG. 2

Thus—

$$Q = Q_0 \left(\theta + \frac{1}{v}\frac{d y}{d t}\right)$$

As before, $S = S_0\, a_0$, the rudder remaining untouched in the position it had during steady motion.

Besides the lateral resistance, there will be a resistance depending on angular velocity. The effect of this resistance will be, that the angular motion is extinguished sooner than would otherwise be the case, but it will not influence the general character of the motion. Even if only the resistance varying as $\frac{d\theta}{d t}$ is taken into account, the discussion of the solution will become very much complicated, and it is, therefore, not included in the general investigation, but deferred to a later section.

The simultaneous differential equations of angular and vertical motion contain, according to the foregoing, second differential co-efficients of θ and y (the vertical ordinate), first differential co-efficient of y and the first power of θ.

The general solution of these equations gives expressions for y and θ, which, besides a constant term, each contain three exponential terms, and the expression for y contains, moreover, a term with t as factor.

The exponents are given by an equation of the third degree, which really determines the character of the motion. It is found that if ϵ and β are both positive, all the exponential terms will vanish, and the motion will be stable. If either ϵ or β are negative, *i.e.*, if the centre of gravity falls above the centre of buoyancy, or, if the centre of lateral resistance falls forward of the centre of gravity, the motion will be unstable ; for then there will always be one real positive exponent, showing an ever increasing motion, or there will be two imaginary exponents with positive real parts, showing ever increasing oscillations.

In the general investigation, ϵ and β are supposed both positive, *i.e.*, stable motion.

The cases $\epsilon = 0$ and $\beta = 0$ are treated separately.

General Character of Disturbed Motion.

Suppose a submarine boat moving with steady motion along a horizontal line, while inclined at an angle θ_0 to the horizon.

If now a disturbance occurs, consisting of a permanent vertical force acting at any point of the axis of the boat, and if the rudder is not touched, she will begin to turn towards a new position of angular equilibrium :

$$\theta_1 = \theta_0 + \frac{f(a + z)}{\epsilon}.$$

This position may be reached either through a series of oscillations, decreasing in amplitude, or through a steady swing.

The limiting deflection is directly proportional to the moment of the disturbing force about the centre of lateral resistance, and inversely proportional to the longitudinal stability.

The centre of gravity will at the same time follow a path, which is at first curved upwards or downwards, possibly undulating, and ending in a straight line inclined to the horizon, showing a limiting vertical velocity :

$$\left(\frac{d\,y}{d\,t}\right)_1 = \frac{f}{Q_0}\,v + \frac{f(a + z)}{\epsilon}\,v.$$

The first of these terms is due to the direct action of the disturbing force taken to act at the centre of lateral resistance; the second term is due to the deflection, it may easily be many times greater than the first term.

It is seen that the vertical limiting velocity is directly proportional to the disturbing force, and to the speed, and that it increases with a decrease in lateral resistance and in longitudinal stability.

Moreover, forces acting forward of the centre of lateral resistance will cause a greater vertical motion than forces acting aft of that point, for the two terms in the expression for $\left(\frac{dy}{dt}\right)_1$ will in the former case be of the same sign, in the latter case of opposite signs.

It may be concluded that it is important to have a good longitudinal stability, and to avoid any change in the longitudinal balance. Movements of weights from aft to forward, and addition of weights forward, are in particular dangerous, because they will cause the boat to descend. Curves 1 and 2 (see Plate) show the path of the centre of gravity when a disturbing downward force is acting forward and aft respectively.

Whether, in the act of passing from the equilibrium of steady motion to the limiting condition, the motion be oscillatory or not, is shown in the Appendix to depend in the first instance on the relation between a certain factor F and the longitudinal stability, ε—

$$F = \frac{Q_0^2 \rho^2}{M v^2}$$

where ρ is radius of gyration and M is the mass of the boat. F depends, therefore, on the shape, size, and speed, and, as Q_0 varies probably nearly as v^2, F will be large in long flat boats with high speed.

The condition that non-oscillatory motion shall be at all possible is found to be—

$$F > 27 \varepsilon.$$

Let us suppose this to be the case, and let us imagine C, the centre of lateral resistance, to move from forward to aft in such a way that only a is varied, but all other elements of the boat, and therefore also F and ε, remain unaltered.

When C is forward of the centre of gravity, G, the motion will be oscillatory and increasing; *i.e.*, the motion is entirely unstable. When C coincides with G, the oscillations will neither increase nor decrease. When C lies aft of G, the oscillations will be decreasing.

As C moves farther aft, the moment of lateral resistance β increases, and therefore

$\gamma = \beta + \varepsilon$ increases. When γ has reached a certain value, the motion will cease to be oscillatory, and the boat will turn steadily from one position of equilibrium to the other.

Moving C further aft, another point will be reached, where the motion becomes again oscillatory, and will continue to be so, however much γ is increased.

Thus there would be three phases in the character of the motion, if C were imagined to be moved from forward to aft.

During the first phase the boat will oscillate, and the oscillations, which will be of great amplitude and long period, are mainly governed by the longitudinal stability, ε. During the second phase the boat will swing steadily to the new position without oscillations. During the third phase the boat will again oscillate, but the oscillations are now governed mainly by the moment of the lateral resistance, and the position of equilibrium will be approached in a series of small oscillations of short period.

For the steering of the submarine boat, it would appear to be desirable to approach or to attain the non-oscillatory condition. We must therefore secure, firstly, the inequality $F > 27\varepsilon$, and, secondly, a sufficiently high value of γ. We shall examine how this may be done in various cases.

It is likely that, in practice, the motion will nearly always be found non-oscillatory, even in such cases where oscillations might be expected to take place. The reason for this is, probably, that the rudder is generally used before the return swing has commenced, and that the angular resistance extinguishes the motion.

High Speed Boats.

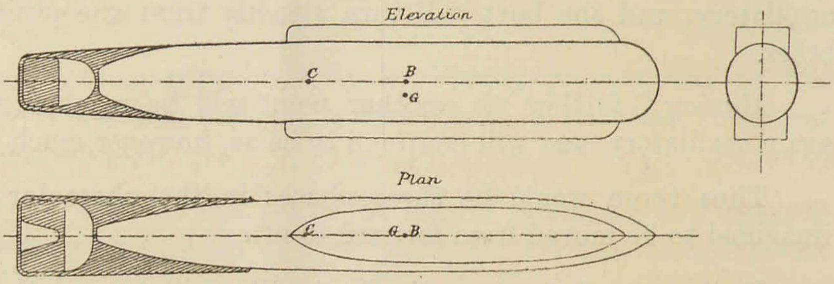
For boats, such as are now generally constructed, of from 100 to 200 tons displacement, and with our present means of under-water propulsion, a speed of above 12 knots must be considered "high speed."

In such boats F will generally be found much greater than 27ε, not only on account of the high value of v, and therefore of Q_0, but also because such boats must be of great length, and therefore of great radius of gyration.

We can therefore afford to make ε large, which is the more desirable in a high-speed boat, because the vertical movements vary mainly as $\frac{v}{\varepsilon}$. But great longitudinal stability means great depth, and as the lateral resistance, Q, is already large in virtue of the speed, we need not make the breadth great.

Thus we arrive at a long, deep, and comparatively narrow boat, with great metacentric height.

Moreover γ must have a certain high value, and therefore a must be made very great by drawing the centre of gravity forward and the centre of lateral resistance aft. This leads to a deep narrow fore-body, a flat and broad aft-body, and may be attained by a shape, such as is indicated diagrammatically on Fig. 3, fitting large horizontal fins aft.



HIGH SPEED BOAT.
FIG. 3.

The French submarine boats, *Gustave Zédé*, *Morse*, and others approach to this type. (*Morse* : $\frac{\text{Length}}{\text{Breadth}} = 13$; speed 12·3 knots).

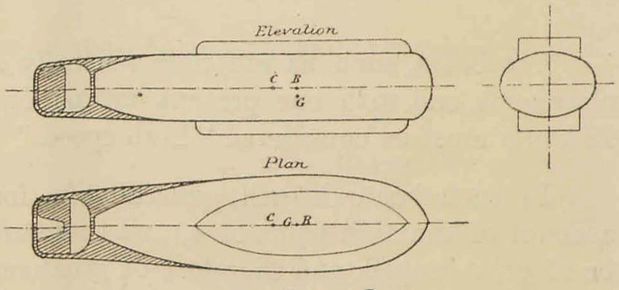
Low Speed Boats.

These may be defined as boats of less than 6 knots.

The longitudinal stability need not be so great in low-speed as in high-speed boats. The inequality, $F > 27\,\epsilon$, may, therefore, be attained by somewhat reducing ϵ. Moreover, F may be increased by making the boat broad and flat.

Great length is not necessary in a low-speed boat, and is, on the whole, objectionable in point of internal arrangements and weight of hull. It will be found that a reduction in ϵ will cause a reduction in the value of γ, for which non-oscillatory motion is reached.

Thus, it is not required in a low-speed boat to draw the centre of lateral resistance so far aft of the centre of gravity as in the high-speed boat. The low-speed boat should, therefore, be of short length, small depth, and great breadth. The shape being given, the metacentric height should be made as great as possible by a low centre of gravity. Large horizontal fins should be placed aft. The general character of the type is indicated on Fig. 4.



LOW SPEED BOAT.
FIG. 4.

The "Holland Torpedo Boat Company's" first boat, the *Holland*, approaches the low-speed type here described $\left(\frac{\text{Length}}{\text{Breadth}} = 5{\cdot}2\text{; speed, 6 to 7 knots}\right)$, but differs in having greater depth and stability.

For doing service off the coasts in deep water, and for semi-offensive purposes, the high-speed type appears best suited. For waters of limited depth and extension, such as estuaries, straits, and sounds, a type between the two extremes would be well adapted. For use in roadsteads and ports, where the depth of water is very limited, where mine-fields make small draught desirable, and where the condition of the sea bottom permits sliding along the bottom when found expedient, the flat, low-speed would appear to be the best.

Effect of Free Liquid Surfaces.

The most likely case of free liquid surfaces occuring, and one which has probably caused the failure of many submarine boats, is that of free water in ballast tanks. The effect is practically the same as if the stability had been diminished. It may be noticed that, in this way, the longitudinal stability may be seriously reduced, while the transverse stability remains practically unaltered; if, namely, the tanks are of small breadth, but of great length.

As the longitudinal stability is of utmost importance for the good behaviour of a submarine boat, the great danger of free liquid surfaces is obvious.

The main ballast tank should, therefore, be subdivided into smaller compartments, and each compartment should, when submerged, be kept either completely filled or completely empty. Two deep end tanks may be used for regulating the trim, and for providing for smaller fluctuations in displacement. The water in these tanks must, unavoidably, have free surfaces, but the length of the tanks may be reduced to a minimum.

Centre of Lateral Resistance Coincides with Centre of Gravity.

Here $\beta = 0$. The angular motion will be stable and purely oscillatory. Were it not for the angular resistance, the oscillations would, in this case, never be extinguished.

Longitudinal Stability Vanished.

Here $\epsilon = 0$ and $\gamma = \beta$. As mentioned, this case may occur when there is free water in ballast tanks, and is of interest also because the equations apply, with few modifications, to motion in a horizontal plane where ϵ is likewise equal to zero. The investigation shows that the motion is always non-oscillatory as long as $\beta < \frac{F}{4}$; but, if the centre of lateral resistance falls more aft, than given by this inequality, *i.e.*, if $a < \frac{Q_0 \rho^2}{4 M v^2}$, the motion will be oscillatory.

The equation for θ shows that θ will be ever increasing with a constant angular velocity if a permanent disturbing force is acting. The equation for y, the vertical ordinate, shows an accelerating vertical velocity. The equations, consequently, hold only during the first stages of the motion, while θ is still small.

The deviations due to a given disturbance are smaller the greater a is, and the greater the speed.

The motion is stable as long as β is positive. If besides $\epsilon = 0$, also $\beta = 0$, the boat will be in a state of neutral equilibrium; θ will vary directly as t^2, and the expression for y will contain terms with t, t^2, and t^3.

For motion in the horizontal plane, *i.e.*, for steering of ordinary vessels, the equations hold almost unaltered, as regards small involuntary deviations from the course, and during the first part of the turning movement, when the rudder is used. β is here referred to the vertical plane.

That stability of motion may exist in case of horizontal steering is evidenced by the Thornycroft torpedo-boats with double spade rudders. These boats may, in calm weather, when going at high speed, keep their course almost unaltered for a quarter of an hour or more, without the rudders being touched. It is probable that the effect of the propeller race on the rudders is such as to make β positive in this case.

If β is negative, as is probably most often the case in the vertical plane in ordinary ships, the motion is unstable, and the rudder must, therefore, generally be used continually in order to keep the ship on her course. The lateral deviations which are hereby incurred are, however, in most cases of small consequence. It is generally immaterial whether the ship moves along one line, or along another parallel to the original at several fathoms' distance.

In narrow passages, where such deviations are not permissible, great care must be exercised in order to keep the ship on the desired track.

A submarine boat is in the vertical sense always, so to speak, manœuvring through a narrow passage; a deviation of a few fathoms upwards may bring it to the surface, where it may be exposed to the attack of the enemy; a deviation downwards may carry it to the bottom, or expose it to crushing pressures.

The case of an ordinary ship turning in a circle with steady velocity is discussed in the Appendix, on account of its analogy to the limiting disturbed motion of a submarine boat.

The equations for initial motion show that when the rudder is placed aft, the centre of gravity will always begin to move out of the course line to the opposite side of that to which the ship is ultimately turning. This holds both in case of vertical and horizontal steering, and is a fact which has often been observed, when laying down the turning circles of ordinary vessels.

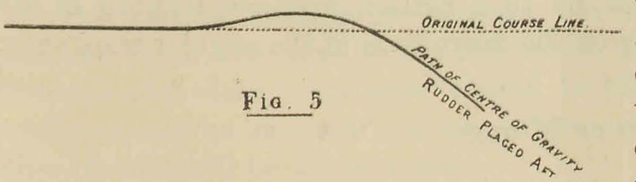


Fig. 5

Impulsive Forces.

By grounding and collision a submarine boat may be subject to forces of impulsive character. Apart from the damage which the boat may suffer, the effect will be that the boat will shift to a different level, where it will proceed in a horizontal line, inclined at the original angle θ_0 to the horizon. Before the boat comes to rest in its new position of equilibrium, it may perform some oscillations, or it may move in a steady double swing.

The case of pure unresisted oscillations, imagined to be produced by impulsive action, is treated in the Appendix; it is of interest for comparison with the more complex case of resisted motion.

Angular Resistance Included.

This resistance must be in character similar to the resistance to rolling of an ordinary vessel, the main point of difference being that the pitching movements of the submarine boat are of much smaller amplitude than the rolling of ordinary vessels.

The late Mr. W. Froude has shown, that at small angles the greater part of the resistance to rolling varies as the first power of $\frac{d\,\theta}{d\,t}$, the smaller part as the second power. It appears, therefore, justifiable to neglect the resistance varying as $\left(\frac{d\,\theta}{d\,t}\right)^2$, and to put the moment of resistance equal to $k\,\frac{d\,\theta}{d\,t}$.

It is found that the form of the expressions for θ and y remains unaltered, and only the co-efficients and the exponents of the vanishing terms change value. The limiting inclination and vertical velocity will, however, be the same as if the resistance $k\,\frac{d\,\theta}{d\,t}$

had not existed, but the limiting condition will be sooner reached, and the oscillatory motion will probably in most cases not be perceptible.

It is found, on account of this resistance, that we may have stable motion even if the centre of lateral resistance is situated, up to a certain distance, forward of the centre of gravity; this distance depending on the magnitude of the angular resistance.

Stability of Motion.

This quality may be defined as the tendency of the boat to preserve the state of steady motion.

It has already been mentioned that, although $\gamma > 0$ is a necessary condition for stability of motion, it is not sufficient. If, namely, one of its component parts, β and ϵ, is negative, the motion will be unstable.

By imagining the boat to be deflected a small angle from its position of steady motion, and then left to itself, it is proved in the Appendix that, even if one of the quantities, β and ϵ, is equal to zero, the other remaining positive, the motion will still be theoretically stable.

Thus the general conditions of stability of motion, disregarding angular resistance, are—

$$\beta \geqq 0, \quad \epsilon \geqq 0 \text{ and } \gamma > 0;$$

i.e., the centre of lateral resistance must not fall forward of the centre of gravity, and the centre of gravity not above the centre of buoyancy. If either of these conditions is not fulfilled, the motion will be unstable.

Although, as in horizontal steering, it may be possible to keep the depth by means of the rudder, even if β is negative, it appears safest, at any rate in experimental boats, to draw the centre of lateral resistance as far aft as possible, and to place the centre of gravity well forward.

This leads to a narrow, deep fore-body, broad and flat aft-body, and large horizontal fins, with rudders placed as far aft as possible. High speed is favourable to stability of motion, but should always be accompanied by great longitudinal stability.

Manœuvring Power.

Manœuvring power may be defined as the power to acquire and to extinguish angular velocity. This quality must be essentially different from the same quality in the horizontal plane, from the existence of the longitudinal stability.

When the vertical rudder is used, the boat acquires a certain angular velocity, and, as long as the rudder acts, it will go on turning indefinitely. But, when the horizontal rudder is used, the movement will, apart from resistances, be limited by the longitudinal stability.

The vertical turning movements which have to be carried out in a submarine boat, are small compared to the horizontal movements of any vessel. Although it is desirable that these movements should be effected quickly, it will, on account of their smallness, not be necessary to use great angles of inclinations, which, moreover, for other reasons will be inconvenient.

Inclinations exceeding 10° will rarely be used. It is clear, therefore, that great angular velocities cannot, and need not, be attained during the manœuvres. The manœuvring power should be secured exclusively by increasing the rudder area, and by placing it far aft; hereby the stability of motion is at the same time somewhat increased. The faculty of steering steadily in a given direction, and of readily extinguishing turning movements, appears more important in a submarine boat than the faculty of quick and easy turning. Stability of motion should, therefore, not be sacrificed in order to obtain manœuvring power. The opposing influence of resistance will be small, and will be rather beneficial, because it will hardly be felt before the rudder is to be eased, and will then assist in bringing the boat to rest in the new direction.

A small moment of inertia is favourable to manœuvring power, but this element must be settled principally with regard to internal arrangements, speed, &c.

Surplus Buoyancy.

It is desirable to retain a certain amount of surplus buoyancy in a submarine boat. It will provide a margin of safety against minor leakages. If the propelling machinery breaks down, the boat will at once ascend to the surface. When on the surface the boat will possess a certain amount of freeboard, without need of pumping. Generally the surplus buoyancy will reside in the conning tower. The surplus buoyancy is not, as some people imagine, kept for the sake of stability; its influence on stability will generally be small, and may even be detrimental: if, namely, it is produced by removing a weight from below the centre of gravity.

The facility with which the boat dives from "awash," must be dependent on the smallness of the angle θ_0, which it has to maintain when in steady motion under water. It is evident that both as regards resistance and convenience this angle should be as small as possible for a given surplus buoyancy, *i.e.*, $\frac{\theta_0}{w}$ should be small.

The investigation shows that this ratio is smallest when the conning tower is placed on the same side of the centre of gravity as the rudder.

Moreover, that the best result is obtained when the rudder is placed forward; and finally, that great central horizontal area and high speed are favourable to a great surplus buoyancy.

The Rudders.

As in any other ship, a powerful steering gear, great area and leverage of rudder, and high speed are elements conducive to good steering qualities in the submarine boat.

As regards the position of the rudders opinions are divided. Some hold that they should be placed forward, others that they should be placed aft, others again both forward and aft, while some inventors combine the fore or aft rudders with midship rudders.

It has already been stated that it is favourable in point of surplus buoyancy and inclination to place the rudders forward. As also forces acting forward produce greater vertical motion than when acting aft, we may conclude that forward rudders must be more effective than aft rudders.

When plunging from the condition "awash," fore rudders are not liable to get out of the water as aft rudders are, and the direct action of the fore rudders will assist the vertical movement, while the upward pressure on aft rudders will counteract the same. It appears, indeed, that with some submarine boats, notably long ones, difficulties have been experienced in effecting the dive using aft rudders only.

In spite of all these arguments in favour of placing the rudders forward, this disposition can hardly be recommended except in very long boats, where it may prove a necessity. The same reasons which, in an ordinary ship, make us place the rudder aft, namely, that it is there better protected against damage and fouling, and that its effect is augmented by the action of the propeller race, hold good in a still higher degree in a submarine boat.

The drawbacks which are connected with placing the rudders aft, may be almost entirely obviated by giving them great area and leverage, and by placing the conning tower aft of the centre of gravity.

It has already been mentioned that stability and steadiness of motion are increased by aft rudders.

Midship rudders produce no rotation, and must, therefore, be combined with either fore or aft rudders. They counteract the surplus buoyancy, and facilitate diving, but their position is very exposed, and they complicate the service of the boat. It appears preferable, therefore, to place rudders aft in all cases, and in very long boats, where experiments might show the necessity thereof, also to place rudders forward; but in such case the forward rudders should be used for steering, the aft rudders only as inclined moveable planes. The combination of four rudders, two forward, and two aft, is used in the French submarine boat *Le Narval*, and probably in several, if not all, other French boats. The American boat, the *Holland*, has only rudders aft.

It has already been mentioned, that steering in the vertical plane is more difficult than in the horizontal plane; the helmsman, who in the latter case is concerned only with the angular motion, is in the submarine boat concerned both with the angular and the vertical motion. Moreover, if a disturbance of permanent nature occurs in a submarine boat, the helmsman has to find the new position of equilibrium both for the boat and for the rudder.

It is obvious that only great experience and practice, and the use of good instruments, can enable the helmsman to solve his task satisfactorily. In an experimental boat a long and careful training of the helmsman, who is to work the horizontal rudders, must, therefore, be an essential condition of success, unless automatic appliances are devised which make his presence superfluous

Effect of Non-Symmetry.

The general conclusion of the discussion on this point is that symmetry should, as far as possible, be preserved, at any rate in an experimental boat, because the effect of non-symmetry will vary with the speed and immersion, and is, therefore, likely to render steering difficult.

If symmetry is not preserved, whether it be in the shape of hull or position of propeller shaft, the vertical component of the resulting extra force should always act downwards, and the force should have the greatest possible turning moment about the stock of the horizontal rudder, opposite, in sign, to that of the surplus buoyancy; hereby the ratio $\frac{\theta_0}{w}$ will be the smallest possible.

Effect of Heeling.

When a submarine boat is inclined transversely, the vertical rudder will act partly as a horizontal one and conversely. Apart from distribution of weights, heeling may be caused by the action of the propeller; this is, in the Whitehead torpedo, avoided by using two screws turning in opposite directions. In a submarine boat, the turning moment of the propeller will, with the present means of under-water propulsion, probably be insignificant.

ON EVEN KEEL.

The principle of keeping the boat always on even keel, not only during steady motion, but also when rising or sinking, has by many inventors, and formerly, when the Whitehead mechanism was yet unknown, also by the author of this paper, been recommended as the safest. The depth may then be kept either by means of inclined planes or fins—probably used in some of the French boats (*Le Narval*)—or by downhaul propellers, or by pumping water in and out of the boat, all of which methods are discussed in the Appendix.

Since the Whitehead mechanism has been made public, there appears to be no necessity to use any of these methods when under way, for what has been done in the Whitehead torpedo, we must also be able to do in a submarine boat, where the pendulum-valve mechanism will act under more favourable conditions than in the torpedo, not being exposed to the violent accelerations of the latter.

We now know that steering has been effected successfully by the inclining method, using horizontal rudders only, namely, in the *Holland*.

The Whitehead mechanism may be used either as an indicator, or for automatically working the rudders through a servo-motor.

For keeping a certain depth at rest under water, a pump may be used. This has been done successfully in *Le Goubet*. When under way, this system does not work well, at any rate in large boats.

The mathematical investigation in the Appendix holds, in these cases, with the modification, that θ_0 is put equal to zero.

STEERING BY SHIFTING WEIGHTS INSIDE THE BOAT.

The shifting of weights may consist in pumping water from one end tank to the other. By this method all danger of damage and fouling of rudders is avoided, and the system appears simple and effective. It is shown in the Appendix that pure couples are very effective in producing vertical motion.

It is doubtful, however, whether sufficiently great turning moments can be created in this way without using a very powerful pump, involving comparatively great expenditure of power.

REGULATING AND BALANCING.

Before diving, the displacement and the trim should be so regulated that, when at rest the boat should lie upright on an even keel, with only that part of the conning tower which corresponds to the surplus buoyancy, projecting above the water. Let this be called "the diving condition." Experience may show that it is advantageous to trim the boat somewhat on the keel in this condition.

Moreover, every precaution should be taken to preserve the displacement and the position of the centre of gravity unaltered during submergence.

The former of these operations, the "balancing," is performed while in the light condition by means of the ballast tanks. It has been stated that there must be at least one large (possibly sub-divided) "bottom tank," and two smaller "end tanks."

Let us suppose that, by experiment, the draught and trim have been determined, from which the boat may be brought directly to the diving condition, by simply filling the bottom tank. Let this be called the "balanced trim," and the condition, the "normal light condition."

By placing the bottom tank well forward, the balanced trim will be such that the boat is well on the keel, the screw well immersed, and the fore-body well out of the water. It is, therefore, a water-line well adapted for navigation.

Suppose now the boat to be approaching the enemy. It should then be kept nearly on its balanced trim until in danger of being discovered. By means of the end tanks the boat should be brought exactly on that trim, whereafter the bottom tank, which has hitherto been kept completely empty, is completely filled. The boat will now be in her "diving condition," ready to dive at any moment.

Before going to dive, it should be ascertained, that the bottom tank is completely filled, that all bilge-water is pumped out, and all loose weights fastened. The boat should be laid exactly upright. Each man should be assigned a certain position, from which he should not move during the dive. The balancing process here described is, in principle, the same operation which must be carried out with every Whitehead torpedo, and for the same reasons.

If, during the dive, a permanent disturbance occurs, it will, within certain limits, be possible to meet it by the action of the rudder alone; the result being that steady motion is again established, but the angle of rudder and the angle of inclination are generally both altered. If the change in rudder angle and inclination is observed to be considerable, it appears advisable either to stop and rebalance the boat, or to keep going and attempt to bring the boat back to the original inclination by pumping water from one end tank to the other.

If after this is done the boat shows a tendency to sink or to rise, it is a sign that a weight has been added or removed, and water should accordingly be pumped either out of or into the boat.

At the present stage of submarine navigation submerged runs of more than a couple of miles will, probably, rarely be undertaken. For such short distances it would appear possible to avoid permanent disturbances of such magnitude, that they could not be dealt with by the rudder alone. The method here described of balancing under way may, therefore, not find any use for the present.

If the sea is too rough for the exact determination of the balanced trim, the balancing operation may, possibly, be performed under water when at rest, by first making the boat float in equilibrium on even keel, and then pumping out a certain amount of water in order to produce the surplus buoyancy.

Summary of Conclusions.

Shape.—The fore-body should be deep, the aft-body broad and flat. Large horizontal fins placed aft. High-speed boats should be longer, deeper, and of smaller breadth than low-speed boats. Symmetry about a horizontal central plane should, as far as possible, be preserved. If non-symmetry is unavoidable, the extra force thereby created should have a vertical component, acting downwards, and the force should have the greatest possible turning moment about the stock of the horizontal rudder.

Stability should, the type being given, be made as great as possible, by securing a low centre of gravity.

The horizontal rudder should be large, and placed as far aft as possible.

Surplus buoyancy should be as great as found convenient. The conning tower should be placed aft of the centre of gravity.

The displacement, and the longitudinal distribution of weights, should be preserved strictly unaltered while submerged, and should be carefully regulated before diving. Unavoidable disturbances should occur aft rather than forward.

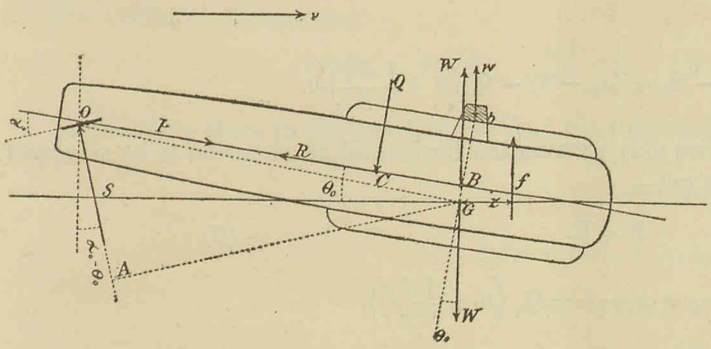
I regret that I have been unable to base this paper more on practical experience and less on theory than has been the case. So far I have not myself had an opportunity of constructing or manœuvring submarine boats, and I have been obliged, therefore, to go entirely by theoretical considerations, and by accounts which have been published regarding past and present submarine boats.

I wish, therefore, to apologise beforehand, if points are found in this paper on which I have laid particular stress, and which may hereafter in practice be proved to be of small importance; and, likewise, if I have treated lightly, or even omitted, points which later experience may show to be of greater importance.

APPENDIX.

Steady Motion.

Referring to the figure: B G must be a small quantity compared to G O, and O will lie practically in the axis, except for very great rudder angles. Drawing G A at right angles to the line of action of S, we may, therefore, approximately, put angle O G A = α. Let G O = p.



In forming the equations of motion, we refer here, and in the following, to a system of rectangular co-ordinates, fixed in space, with origin in the initial position of the centre of gravity of the boat.

The axis of X is horizontal and positive to the right. The axis of Y, vertical and positive upwards. The angles are measured from the positive direction of the axis of X, positive in direction of the hands of a watch.

Resolving, horizontally—

$$M \frac{d^2 x}{d t^2} = P - R - S_0 \alpha_0 (\alpha_0 - \theta_0) - Q_0 \theta_0^2 = 0.$$

Neglecting squares of the small angles, we find—

$$P = R. \qquad (1)$$

Resolving, vertically—

$$M \frac{d^2 y}{d t^2} = S_0 \alpha_0 + w - Q_0 \theta_0 = 0. \qquad (2)$$

Taking moments about G—

$$M \rho^2 \frac{d^2 \theta}{d t^2} = S_0 p \alpha_0 - Q_0 \theta_0 a - W . BG . \theta_0 + (P - R) . BG - w . Gb . \theta_0 = 0.$$

As w is a small quantity, we may neglect the last term, and, substituting, we get—

$$S_0 p \alpha_0 = \theta_0 (\beta + \epsilon) = \theta_0 \gamma. \qquad (3)$$

From (2) and (3) we find—

$$\theta_0 = \frac{w p}{\delta} \qquad (4)$$

and—

$$\alpha_0 = \frac{w \gamma}{S_0 \delta} \qquad (5)$$

where—

$$\delta = Q_0 (p - a) - W \times BG.$$

The turning moment being $S_0 p \alpha_0 - \theta \gamma$, we see that any increase in θ will cause the turning moment to decrease as long as γ is positive, but to increase if γ is negative. Conversely for a decrease in θ. $\gamma > 0$ is, therefore, a necessary condition of stability of motion.

Disturbed Motion.

Formation and Solution of the Equations of Motion.

The boat is supposed to be in steady motion, when at time $t = o$, a small vertical force f begins to act upwards (positive), at a distance z forward of G, thus producing a negative turning moment. The rudder is not touched, and remains at the angle α_0.

Resolving, horizontally—

$$M\frac{d^2 x}{d t^2} = P - R - S_0\,\alpha_0\,(\alpha_0 - \theta) - Q_0\left(\theta + \frac{1}{v}\frac{d y}{d t}\right)\theta.$$

Considering only small disturbances, we may assume the horizontal acceleration to be zero, and, neglecting small terms of second order, we get—

$$P = R. \qquad (6)$$

Resolving, vertically—

$$M\frac{d^2 y}{d t^2} = S_0\,\alpha_0 + w + f - Q_0\left(\theta + \frac{1}{v}\frac{d y}{d t}\right).$$

From steady motion—

$$S_0\,\alpha_0 + w - Q_0\,\theta_0 = 0.$$

Subtracting—

$$\frac{d^2 y}{d t^2} + \frac{Q_0}{M v}\frac{d y}{d t} + \frac{Q_0}{M}\theta = \frac{Q_0\,\theta_0 + f}{M}. \qquad (7)$$

Taking moments about G—

$$M\rho^2\frac{d^2\theta}{d t^2} = S_0 p\,\alpha_0 - \beta\left(\theta + \frac{1}{v}\frac{d y}{d t}\right) - \theta\varepsilon + (P - R)\,BG - w\,.\,G b\,\theta - f z.$$

Neglecting small terms, and substituting—

$$\frac{d^2\theta}{d t^2} + \theta\frac{\gamma}{M\rho^2} + \frac{\beta}{M\rho^2 v}\frac{d y}{d t} = \frac{\gamma\,\theta_0 - f z}{M\rho^2}. \qquad (8)$$

The general solution of the two simultaneous differential equations (7) and (8) is found to be—

$$\theta = A_0 + A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t} + A_3 e^{\lambda_3 t} \qquad (9)$$

$$y = B_0 + B_1 e^{\lambda_1 t} + B_2 e^{\lambda_2 t} + B_3 e^{\lambda_3 t} + \left[\frac{f}{Q_0} + \frac{f(a+z)}{\varepsilon}\right] v\,t. \qquad (10)$$

Where the A's and B's are constants, determined by the initial conditions, and by the mutual relations which exist between them, and where λ_1, λ_2, λ_3 are roots in the equation—

$$\lambda^3 + \lambda^2\frac{Q_0}{M v} + \lambda\frac{\gamma}{M\rho^2} + \frac{\varepsilon\,Q_0}{M^2\rho^2 v} = 0. \qquad (11)$$

Determination of Constants.

By substitution in (8) from (9) and (10), we find—

$$A_0 = \theta_0 - f\frac{a+z}{\varepsilon}$$

$$B_1 = -A_1 \frac{\lambda_1^2 M \rho^2 v + \gamma v}{\beta \lambda_1} \tag{13}$$

$$B_2 = -A_2 \frac{\lambda_2^2 M \rho^2 v + \gamma v}{\beta \lambda_2} \tag{14}$$

$$B_3 = -A_3 \frac{\lambda_3^2 M \rho^2 v + \gamma v}{\beta \lambda_3}. \tag{15}$$

The initial conditions are:—

$$t = 0, \quad \theta = \theta_0, \quad \frac{d\theta}{dt} = 0, \quad y = 0, \quad \frac{dy}{dt} = 0,$$

Substituting these in (9) and (10), we find the following relations between the constants—

$$A_1 + A_2 + A_3 = f\frac{a + z}{\varepsilon} \tag{16}$$

$$A_1 \lambda_1 + A_2 \lambda_2 + A_3 \lambda_3 = 0 \tag{17}$$

$$A_1 \lambda_1^2 + A_2 \lambda_2^2 + A_3 \lambda_3^2 = -\frac{f z}{M \rho^2} \tag{18}$$

$$\frac{A_1}{\lambda_1} + \frac{A_2}{\lambda_2} + \frac{A_3}{\lambda_3} = \frac{\beta}{\gamma v} B_0. \tag{19}$$

We have thus four equations for the determination of A , A_2, A_3, and B_0.

Put—

$$f\frac{a + z}{\varepsilon} = K$$

and—

$$-\frac{f z}{M \rho^2} = L$$

we then find, by solving the equations—

$$A_1 = \frac{L + K \lambda_2 \lambda_3}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)} \tag{20}$$

$$A_2 = \frac{L + K \lambda_1 \lambda_3}{(\lambda_2 - \lambda_3)(\lambda_2 - \lambda_1)} \tag{21}$$

$$A_3 = \frac{L + K \lambda_1 \lambda_2}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)} \tag{22}$$

and—

$$B_0 = \frac{\gamma v}{\beta}\left(\frac{A_1}{\lambda_1} + \frac{A_2}{\lambda_2} + \frac{A_3}{\lambda_3}\right). \tag{23}$$

Thus all the constants may be determined when the λ exponents are known.

It will be noticed that the disturbing force occurs as a factor in all the constants, while the exponents are independent of this force.

Determination of Exponents.

Returning to the equation (11)—

$$\lambda^3 + \lambda^2 \frac{Q_0}{M v} + \lambda \frac{\gamma}{M \rho^2} + \frac{\varepsilon Q_0}{M^2 \rho^2 v} = 0.$$

Put—

$$\lambda = x - \frac{Q_0}{3 M v}$$

and the equation reduces to—

$$x^3 + x A + B = 0$$

where—

$$A = \frac{\gamma}{M \rho^2} - \frac{Q_0^2}{3 M^2 v^2}$$

and—

$$B = \frac{2}{27} \frac{Q_0^3}{M^3 v^3} - \frac{Q_0 \gamma}{3 M^2 \rho^2 v} + \frac{\varepsilon Q_0}{M^2 \rho^2 v}.$$

The roots of this equation are—

$$x_1 = p_1 + q_1$$
$$x_2 = p_1 \alpha_1 + q_1 \alpha^2$$
$$x_3 = p_1 \alpha^2 + q_1 \alpha$$

where—

$$\alpha = -\frac{1}{2} + \frac{\sqrt{-3}}{2}$$

$$\alpha^2 = -\frac{1}{2} - \frac{\sqrt{-3}}{2}$$

and p_1 and q_1 are determined by—

$$p_1^3 = -\frac{B}{2} + \sqrt{\frac{B^2}{4} + \frac{A^3}{27}}$$

$$q_1^3 = -\frac{B}{2} - \sqrt{\frac{B^2}{4} + \frac{A^3}{27}}$$

and, therefore—

$$\lambda_1 = p_1 + q_1 - \frac{Q_0}{3 M v} \qquad (24)$$

$$\lambda_2 = p_1 \alpha + q_1 \alpha^2 - \frac{Q_0}{3 M v} \qquad (25)$$

$$\lambda_3 = p_1 \alpha^2 + q_1 \alpha - \frac{Q_0}{3 M v} \qquad (26)$$

from which the three exponents may be determined.

As regards the character of these exponents, we see directly, from (11), that as long as γ is positive, which has been shown to be necessary for stable motion, all real roots must be negative. We know, moreover, that in an equation of the third degree there will be either one real and two imaginary roots, namely, when the discriminant, $\frac{B^2}{4} + \frac{A^3}{27}$ is positive; or three real roots, of which two are equal, namely, if the discriminant is zero, and three real and unequal roots, if it is negative.

Examination of the Discriminant.

$$\frac{B^2}{4} + \frac{A^3}{27} = \frac{1}{27 M^3 \rho^6} \left[\gamma^3 - \frac{F}{4} \gamma^2 - \frac{9 F \varepsilon}{2} \gamma + \frac{27 F \varepsilon^2}{4} + F \varepsilon^2 \right]$$

putting—

$$\frac{Q_0^2 \rho^2}{M v^2} = F.$$

The discriminant has the same sign as the expression inside the brackets, which may be denoted D. Thus—

$$D = \gamma^3 - \frac{F}{4}\gamma^2 - \frac{9F\epsilon}{2}\gamma + \frac{27F\epsilon^2}{4} + F^2\epsilon \qquad (27)$$

F depends on shape, size, and distribution of weights, and on the speed, and is always positive. If we take Q_0 to vary as v^2, F will vary as v^2. Both ϵ and γ are likewise positive.

We shall now examine how D varies with variation of γ, keeping ϵ unaltered, *i.e.*, by imagining the centre of lateral resistance to move.

When γ is negative, D will begin at $-\infty$; and, as γ decreases towards zero, there must be a certain negative, real value of γ, for which D is zero. When γ is zero, D will be positive, and equal to $\frac{27F\epsilon^2}{4} + F^2\epsilon$. As $D = 0$ is an equation of third degree in γ, we know that, besides the negative value of γ, already mentioned, there must be two positive real, or two imaginary, values of γ, which make $D = 0$.

Increasing γ positively, D will, therefore, decrease to a certain minimum, which may or may not be negative, according as the two remaining roots of $D = 0$ are real or imaginary.

Hereafter, D increases in the positive direction, and when $\gamma > \frac{F}{3}$, in which case A is positive, D can never become negative. Increasing γ beyond this, D will increase to $+\infty$. By differentiation of D, we find that the maximum and minimum values of D are given by—

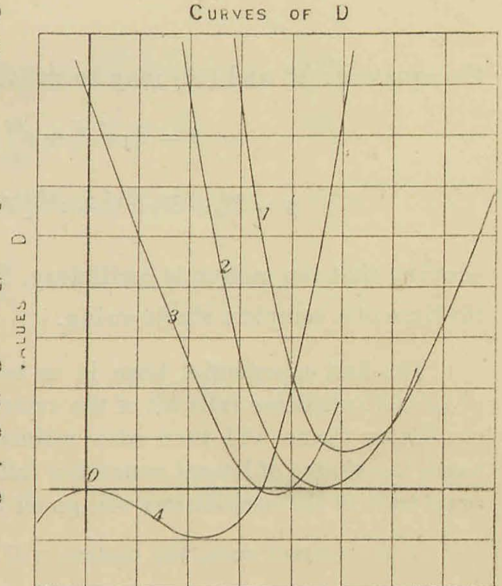
$$\gamma = \frac{F}{12} \pm \sqrt{\frac{F^2}{144} + \frac{3}{2}F\epsilon}$$

showing that D has a minimum on the positive side and a maximum on the negative side.

The general character of the variation of D is best shown by means of curves drawn on values of γ as abcissæ.

The diagram, Fig. 6, shows four curves: (1) giving one negative and two imaginary roots of the equation $D = 0$; (2) one negative and two positive equal roots; (3) one negative and two positive unequal roots; and (4) one positive and two zero roots. We shall examine each of these cases separately, Case (4), later on, being a special case.

CURVES OF D



VALUES OF γ.

F IS THE SAME FOR ALL CURVES

1. F - 19 ϵ
2. F - 27 ϵ
3. F - 56 ϵ
4. ϵ - 0

FIG. 6.

Case (1).

Here D is positive for all positive values of γ, wherefore the equation for λ will have one real root, known to be negative, and two imaginary roots, generally consisting of a real and an imaginary part. Let the three roots be—

$$\lambda_1, \quad \lambda_2 = s + iu, \quad \text{and} \quad \lambda_3 = s - iu$$

therefore—

$$(\lambda - \lambda_1)(\lambda - s - i u)(\lambda - s + i u) = 0\,;$$

which, taken together with equation (11), gives—

$$\frac{Q_0}{M v} = -(\lambda_1 + 2 s)$$

$$\frac{\gamma}{M \rho^2} = s^2 + u^2 + 2 \lambda_1 s$$

$$\frac{\epsilon Q_0}{M^2 \rho^2 v} = -\lambda_1 (s^2 + u^2)\,;$$

from which—

$$s^3 + s^2 \frac{Q_0}{M v} + \frac{1}{4} s \left[\frac{Q_0^2}{M^2 v^2} + \frac{\gamma}{M \rho^2}\right] + \frac{Q_0 \beta}{8 M^2 \rho^2 v} = 0,$$

showing that $s = 0$ when $\beta = 0$, *i.e.*, $\gamma = \epsilon$, and for no other real value of γ. When β is negative and small, s becomes positive. When β is positive, s will be negative.

If we put—

$$A_2 + A_3 = L_2$$

$$i (A_2 - A_3) = L_3$$

$$B_2 + B_3 = M_2$$

$$i (B_2 - B_3) = M_3\,;$$

the equations (9) and (10) may be written—

$$\theta = A_1 e^{\lambda_1 t} + e^{s t} [L_2 \cos u t + L_3 \sin u t] + \theta_0 - f \frac{a + z}{\epsilon} \tag{28}$$

$$y = B_0 + B_1 e^{\lambda_1 t} + e^{s t} [M_2 \cos u t + M_3 \sin u t] + \left[\frac{f}{Q_0} + \frac{f (a + z)}{\epsilon}\right] v t \tag{29}$$

showing that the motion is oscillatory, the period of the oscillation being given by $T = \frac{\pi}{u}$, where T is the time of a complete single swing.

The first exponential term is in both equations evanescent, since λ_1 is negative. If the centre of lateral resistance falls aft of the centre of gravity, we have seen that s will be negative, and the oscillatory terms will then also ultimately vanish, *i.e.*, the motion will be stable. If, on the other hand, the centre of lateral resistance falls forward of the centre of gravity, s will be positive, and the amplitude of the oscillations will go on increasing, *i.e.*, the motion will be unstable.

In the former case, the motion will tend towards a limiting condition, given by—

$$\theta_1 = \theta_0 - f \frac{a + z}{\epsilon} \tag{30}$$

$$y = B_0 + \left[\frac{f}{Q_0} + \frac{f (a + z)}{\epsilon}\right] v t \tag{31}$$

giving a limiting vertical velocity—

$$\left(\frac{d y}{d t}\right)_1 = \left[\frac{f}{Q_0} + \frac{f (a + z)}{\epsilon}\right] v. \tag{32}$$

Case (2).

Here the curve of D touches the axis of abcissæ, giving $D = 0$ for two positive equal values of γ. This is the dividing condition for oscillatory and non-oscillatory motion, and it is, therefore, of interest to know the relations which in this case exist between ϵ and F.

The equation $D = 0$, having two equal roots at this point, its discriminant D' must here be zero. It will be found that $D' = 0$, when—

$$\overline{27}^3 \times \epsilon^3 - 3 \times \overline{27}^2 F \epsilon^2 + 3 \times 27 F^2 \epsilon - F^3 = 0,$$

and this is satisfied only when $F = 27\,\epsilon$. The roots of the equation $D = 0$ are then found to be—

$$\gamma = 9\,\epsilon, \quad \gamma = 9\,\epsilon, \quad \text{and} \quad \gamma = -\frac{45}{4}\,\epsilon.$$

Thus—

$$\gamma = 9\,\epsilon = \frac{F}{3}$$

is the point where the curve of D touches the axis of abcissæ.

The equation for λ gives—

$$\lambda_1 = \lambda_2 = \lambda_3 = -\frac{Q_0}{3\,M\,v}$$

all roots real, negative, and equal.

The equations (9) and (10) become—

$$\theta = e^{-\frac{Q_0}{3Mv}t}\left[A_1 + A_2\,t + A_3\,t^2\right] + \theta_0 - f\,\frac{(a+z)}{\epsilon} \qquad (33)$$

$$y = B_0 + e^{-\frac{Q_0}{3Mv}t}\left[B_1 + B_2\,t + B_3\,t^2\right] + \left[\frac{f}{Q_0} + f\,\frac{(a+z)}{\epsilon}\right] v\,t. \qquad (34)$$

The motion is non-oscillatory, tending towards the same limiting condition as in Case 1. In this special case we have—

$$F = 3\,\gamma, \qquad \beta = 8\,\epsilon, \qquad \frac{Q_0{}^2\,\rho^2}{M\,v^2} = 27\,\epsilon,$$

$$\frac{a^2}{\rho^2} = \frac{64}{27}\,\frac{g \times BG}{v^2},$$

where g is the acceleration of gravity.

Case (3).

If the expression for D' is differentiated with respect to F, and if, in the result, we put $F = (27 + n)\,\epsilon$, it will be found that—

$$\frac{d\,D'}{d\,F} = -\frac{(27+n)\,\epsilon^5}{27 \times 64}\,(81\,n^2 + 5\,n^3)$$

showing that, if we increase F a little beyond the particular value, $F = 27\,\epsilon$, D' will become negative, showing that the equation $D = 0$ will then have three real roots, γ. The curve of D will, therefore, cut the axis of abcissæ in two points on the positive side, and between the values of γ, corresponding to these points, D will be negative. This is Case (3).

A decrease in F (n negative) makes D′ positive, *i.e.*, one real and two imaginary roots of $D = 0$. This is Case (1).

Thus, the condition that Case (3) shall at all exist is $F > 27\epsilon$. If this is not the case, the motion will always be oscillatory; and, if it is the case, and γ at the same time has such a value that D is negative, we shall have three real and negative values of the exponents λ, and the motion will be non-oscillatory.

The equations for θ and y become—

$$\theta = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t} + A_3 e^{\lambda_3 t} + \theta_0 - f\frac{a+z}{\epsilon} \qquad (35)$$

$$y = B_0 + B_1 e^{\lambda_1 t} + B_2 e^{\lambda_2 t} + B_3 e^{\lambda_3 t} + \left[\frac{f}{Q_0} + \frac{f(a+z)}{\epsilon}\right] v t. \qquad (36)$$

The exponential terms are, in this case, always evanescent, and the limiting conditions are the same as in Cases (1) and (2).

In the two special cases where γ has such a value that $D = 0$, *i.e.*, the points where the curve of D cuts the axis of abcissæ, we find—

$$p_1 = q_1 = -\sqrt[3]{\frac{B}{2}}$$

$$\lambda_1 = -2\sqrt[3]{\frac{B}{2}} - \frac{Q_0}{3Mv}$$

$$\lambda_2 = \lambda_3 = \sqrt[3]{\frac{B}{2}} - \frac{Q_0}{3Mv}.$$

$$\theta = A_1 e^{\lambda_1 t} + e^{\lambda_2 t}[A_2 + A_3 t] + \theta_0 - f\frac{a+z}{\epsilon} \qquad (37)$$

$$y = B_0 + B_1 e^{\lambda_1 t} + e^{\lambda_2 t}[B_2 + B_3 t] + \left[\frac{f}{Q_0} + \frac{f(a+z)}{\epsilon}\right] v t. \qquad (38)$$

The motion is here non-oscillatory; but for all other values of γ, lower or higher than those for which D is zero or negative, the motion will be oscillatory. If $\gamma > \frac{F}{3}$ the motion will always be oscillatory.

It will be found that, with small values of ϵ, the inequality $F > 27\epsilon$ may be readily secured, and, probably, also, the value of γ necessary to make D negative. But, if ϵ is very great, we can only make $F > 27\epsilon$ by high speed; and it will be found that, with the high value of F, γ and, therefore, β must have a very high value, in order to make D negative. In other words, regarding it as desirable to make the motion non-oscillatory, the centre of lateral resistance should be drawn farther aft in a boat with great longitudinal stability, than in a boat with small longitudinal stability.

Limiting Conditions.

We have seen that whether, or not, the boat in stable motion oscillates, during the first period, after a disturbance has occurred, she will ultimately settle down to a limiting state—

$$\theta_1 = \theta_0 - \frac{f(a+z)}{\varepsilon}$$

$$\left(\frac{dy}{dt}\right)_1 = \frac{f}{Q_0} v + \frac{f(a+z)}{\varepsilon} v$$

The first term of the limiting vertical velocity follows the sign of the disturbing force ; the second is of sign opposite to that of its moment about C.

The term $\frac{fz}{\varepsilon}$ is the inclination produced by fz, if the boat were at rest. This angle is, in the following, denoted by ϕ on several occasions.

If f is acting at the centre of lateral resistance, we have—

$$\theta_1 = \theta_0 \text{ and } \left(\frac{dy}{dt}\right)_1 = \frac{f}{Q_0} v$$

i.e., the boat will settle down to the original inclination, but will acquire a steady vertical velocity.

If $z = -\frac{\gamma}{Q_0}$, we get—

$$\theta_1 = \theta_0 + \frac{f}{Q_0}, \quad \left(\frac{dy}{dt}\right)_1 = 0, \quad \text{and } y_1 = B_0$$

i.e., the boat suffers a certain angular deflection, but will, in the limit, proceed along a horizontal line, lying a distance B_0 from the original level.

If f is acting at G, $z = 0$, and we get—

$$\theta_1 = \theta_0 - \frac{fa}{\varepsilon}, \quad \left(\frac{dy}{dt}\right)_1 = \frac{f}{Q_0} v + \frac{fa}{\varepsilon} v.$$

This may occur, for instance, if the boat passes through water of varying density.

If an internal weight f is moved through a distance z inside the boat, a pure couple fz is produced. If the weight is moved from forward to aft the limiting conditions become—

$$\theta_1 = \theta_0 - \frac{fz}{\varepsilon}$$

and—

$$\left(\frac{dy}{dt}\right)_1 = \frac{fz}{\varepsilon} v.$$

Already, when $z = +\frac{\epsilon}{Q_0}$, this disturbance will equal that caused by adding a weight at C, but z may easily exceed this quantity many times. Hence, the shifting of a certain weight is generally more dangerous than the addition or removal of the same weight at the middle of the boat.

Initial Motion.

The acceleration is, initially—

$$\frac{d^2\theta}{dt^2} = -\frac{fz}{M\rho^2}$$

and—

$$\frac{d^2 y}{d t^2} = \frac{f}{\mathrm{M}}.$$

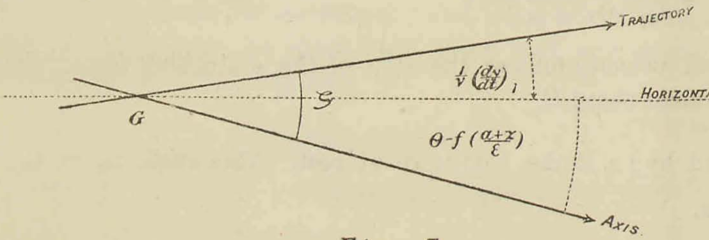


FIG 7

It is of interest to notice that, if f is acting aft, the path of the centre of gravity will at first swerve out to the side opposite to that to which it is ultimately turning. The same phenomenon is observed in the turning circles of ordinary vessels, where the rudder is placed aft. (See Fig. 5.)

Justification for reckoning the Velocity along the Trajectory equal to Original Velocity, v.

The angle ζ, between the axis of the boat and the tangent to the limiting trajectory of the centre of gravity, is—

$$\zeta = \theta_0 - f\frac{a+z}{\varepsilon} + \frac{1}{v}\left(\frac{d y}{d t}\right)_1 = \theta_0 + \frac{f}{\mathrm{Q}_0}$$

and differs, therefore, from that of steady motion only by the angle $\frac{f}{\mathrm{Q}_0}$, which it appears justifiable to neglect as long as f is small. Thus the resistance, and, therefore, the speed along the trajectory, will remain practically unaltered.

Effect of Free Liquid Surfaces.

Suppose a rectangular tank of length l and breadth b to be partly filled with water, then by longitudinal inclination a moment will be created $= \frac{1}{420} b\, l^3\, \theta = k\, \theta$ ft. ts.

Working this moment in the equations of motion, we find that the effect is precisely the same as if ϵ had been reduced by a quantity k. The limiting conditions are—

$$\theta_1 = \theta_0 - f\,\frac{a+z}{\varepsilon - k}$$

$$\left(\frac{d y}{d t}\right)_1 = \frac{f}{\mathrm{Q}_0}\,v + f\,\frac{a+z}{\varepsilon - k}\,v$$

and the exponents are determined by—

$$\lambda^3 + \lambda^2\,\frac{\mathrm{Q}_0}{\mathrm{M}\,v} + \lambda\,\frac{\gamma - k}{\mathrm{M}\,\rho^2} + \frac{\mathrm{Q}_0\,(\varepsilon - k)}{\mathrm{M}^2\,\rho^2\,v} = 0.$$

If $k = \epsilon$ the motion becomes unstable.

Centre of Lateral Resistance Coincides with Centre of Gravity.

Here $a = 0$, therefore $\beta = 0$ and $\gamma = \epsilon$. The equations of motion become—

$$\frac{d^2 y}{d t^2} + \frac{Q_0}{M v}\frac{d y}{d t} + \frac{Q_0}{M}\theta = \frac{Q_0 \theta_0 + f}{M}$$

$$\frac{d^2 \theta}{d t^2} + \theta \frac{\varepsilon}{M \rho^2} = \frac{\theta_0 \varepsilon - f z}{M \rho^2}.$$

In this case $D = \epsilon^3 + 2 F \epsilon^2 + F^2 \epsilon$, which is always positive. The motion will therefore always be oscillatory. The exponents are—

$$\lambda_1 = -\frac{Q_0}{M v}, \quad \lambda_2 = + i\sqrt{\frac{\varepsilon}{M \rho^2}}, \quad \lambda_3 = - i\sqrt{\frac{\varepsilon}{M \rho^2}}.$$

Moreover—

$$s = 0, \quad A_1 = 0, \quad A_2 = A_3 = \frac{K}{2}, \quad L_2 = K, \quad L_3 = 0.$$

Put—

$$\frac{f z}{\varepsilon} = \phi \text{ and } T = \frac{\pi}{\sqrt{\frac{\varepsilon}{M \rho^2}}},$$

then the solution of the equations of motion is—

$$\theta - \theta_0 = -\phi\left[1 - \cos\frac{\pi}{T}t\right]$$

$$y = B_0 + B_1 e^{-\frac{Q_0}{M v}t} + M_2 \cos\frac{\pi t}{T} + M_3 \sin\frac{\pi t}{T} + \left[\frac{f}{Q_0} + \phi\right] v t.$$

The angular motion is one of pure unresisted oscillations, the boat oscillating between the positions θ_0 and $\theta_0 - 2\phi$. Ultimately the oscillations would in reality be extinguished by the resistance, varying with angular velocity, and the limiting position would be $\theta_1 = \theta_0 - \phi$.

The vertical motion will be oscillatory, but will ultimately settle down to—

$$\left(\frac{d y}{d t}\right)_1 = \left[\frac{f}{Q_0} + \phi\right] v.$$

Longitudinal Stability Vanished.

Here $\epsilon = 0$, and, therefore, $\gamma = \beta$

$$\theta_0 = \frac{w p}{Q_0 (p - a)} = \frac{S_0 p a_0}{\beta}$$

$$\frac{d^2 \theta}{d t^2} + \frac{\beta}{M \rho^2}\theta + \frac{\beta}{M \rho^2 v}\frac{d y}{d t} = \frac{\theta_0 \beta - f z}{M \rho^2}$$

$$\frac{d^2 y}{d t^2} + \frac{Q_0}{M v}\frac{d y}{d t} + \frac{Q_0}{M}\theta = \frac{Q_0 \theta_0 + f}{M}$$

The solution is—

$$\theta = \theta_0 - \frac{f z}{\beta} + \frac{f(z + a)\rho^2}{M v^2 a^2} - \frac{f(z + a)}{M a v} t + A_2 e^{\lambda_2 t} + A_3 e^{\lambda_3 t}$$

$$y = \frac{f(z + a)\rho^4}{M a^3 v^2} - \frac{f z \rho^2}{\beta a} - \frac{f(z + a)\rho^2}{M v a^2} t + \frac{1}{2}\frac{f(z + a)}{M a} t^2 + B_2 e^{\lambda_2 t} + B_3 e^{\lambda_3 t}$$

where—

$$\lambda_{2,3} = -\frac{Q_0}{2\,M\,v} \pm \sqrt{\frac{Q_0^2}{4\,M^2\,v^2} - \frac{\beta}{M\,\rho^2}}.$$

As long as—

$$\beta \leq \frac{Q_0^2\,\rho^2}{4\,M\,v^2},\ i.e., \leq \frac{F}{4}$$

the roots λ_2 and λ_3 will be real, and the motion non-oscillatory. The exponents being negative, the exponential terms will be evanescent. If $\beta > \frac{F}{4}$, the motion will be oscillatory.

Curve (4) on Fig. 6 shows the values of D in this case, corresponding to various values of β.

Applying the equations to an ordinary ship turning horizontally, we have $\theta_0 = 0$. If the centre of lateral resistance lies aft of the centre of gravity, *i.e.*, a positive (a case which probably occurs in certain torpedo boats), then the equations for θ and y, given above, will hold, if we only put $\theta_0 = 0$, and make z negative.

If we suppose $a = 0$, and, therefore, $\beta = 0$, a case which is very likely to be approached by ordinary ships, we get—

$$\frac{d^2\,\theta}{d\,t^2} = \frac{f\,z}{M\,\rho^2},$$

$$\frac{d^2 y}{d\,t^2} + \frac{Q_0}{M\,v}\frac{d\,y}{d\,t} + \frac{Q_0}{M}\,\theta = \frac{f}{M};$$

from which—

$$\theta = \frac{f\,z}{2\,M\,\rho^2}\,t^2,$$

and—

$$y = \left[\frac{M\,v^2 f}{Q_0^2} - \frac{M^2\,v^4\,f\,z}{\rho^2\,Q_0^3}\right] e^{-\frac{Q_0}{M\,v}t} - \frac{f\,z\,v}{6\,M\,\rho^2}\,t^3 + \frac{f\,z\,v^2}{2\,Q_0\,\rho^2}\,t^2 + \left[\frac{v\,f}{Q_0} - \frac{v^3\,M\,f\,z}{\rho^2\,Q_0^2}\right] t.$$

These equations hold good, however, only during the first stages of the turning, while θ is still small.

The limiting trajectory of an ordinary ship is known to be a circle, in which the ship turns with steady velocity, her head turned inwards, the axis forming a certain angle—the "drift angle"—with the tangent to the turning circle.

This case is analogous to the steady limiting motion of the submarine boat, only the trajectory is in the latter a straight line, in the ship a circle. The "drift angle" corresponds to ζ. The force acting on the rudder is: $S_0 \sin a_0 \cos a_0 = f$. Let r be the radius of the turning circle, then one equation of motion will be—

$$\frac{M\,v^2}{r} = Q_0\,\zeta - f$$

or—

$$\zeta = \frac{\frac{M\,v^2}{r} + f}{Q_0} = \text{drift angle.}$$

The turning moment is balanced by the resistance, which may be taken in this case approximately to be $W\,k\left(\frac{d\,\theta}{d\,t}\right)^2$.* The angular acceleration is zero, and the angular velocity therefore constant, let it be ω.

Thus—

$$f\,p - W\,k\,\omega^2 - Q_0\,\zeta\,a = 0$$

or—

$$k\,W\,\omega^2 = f\,(p - a) - \frac{M\,a\,v^2}{r}.$$

Not knowing the law according to which Q_0, k, and a vary, these formulæ can, however, hardly be turned to any practical use in the present state of our knowledge.

Impulsive Forces.

Let an upward impulsive force P be acting forward, and at a distance z from G.

Let the impulse create an angular velocity ω and a vertical velocity v_1, then—

$$P = M\,v_1 \text{ and } \omega = -\frac{P\,z}{M\,\rho^2}$$

or—

$$v_1 = -\,\omega\,\frac{\rho^2}{z}.$$

The equations become—

$$\theta = A_1\,e^{\lambda_1 t} + A_2\,e^{\lambda_2 t} + A_3\,e^{\lambda_3 t} + \theta_0$$

$$y = B_0 + B_1\,e^{\lambda_1 t} + B_2\,e^{\lambda_2 t} + B_3\,e^{\lambda_3 t}$$

the motion being oscillatory, or not, according to the value of D.

With the altered initial conditions, we get—

$$A_1 + A_2 + A_3 = 0$$

$$A_1\,\lambda_1 + A_2\,\lambda_2 + A_3\,\lambda_3 = -\,\omega$$

$$B_0 + B_1 + B_2 + B_3 = 0$$

$$A_1\,\lambda_1^2 + A_2\,\lambda_2^2 + A_3\,\lambda_3^2 = \frac{\beta\,v_1}{M\,\rho^2\,v} = L,$$

from which—

$$A_1 = \frac{L + \omega\,(\lambda_2 + \lambda_3)}{(\lambda_1 - \lambda_3)\,(\lambda_1 - \lambda_2)}$$

and similarly for A_2 and A_3.

The limiting conditions are—

$$\theta_1 = \theta_0, \quad \left(\frac{d\,y}{d\,t}\right)_1 = 0, \quad y_1 = B_0 = \frac{P\,z\,v}{\beta}.$$

* Manual of Naval Architecture, by W. H. White, 1882, page 615.

Pure Unresisted Oscillations.

If the disturbing force is impulsive the differential equations are—

$$\frac{d^2\,\theta}{d\,t^2} + \frac{\gamma}{M\,\rho^2}\,\theta = \frac{\gamma\,\theta_0}{M\,\rho^2}$$

$$\frac{d^2\,y}{d\,t^2} + \frac{Q_0}{M}\,\theta = \frac{Q_0\,\theta_0}{M}.$$

The general solution of the first equation is—

$$\theta = L_1 \cos t \sqrt{\frac{\gamma}{M\,\rho^2}} + L_2 \sin t \sqrt{\frac{\gamma}{M\,\rho^2}} + \theta_0.$$

Suppose now that at time $t = 0$ the boat has received an upward impulse acting aft, which has created a vertical velocity—

$$v_1 = \frac{Q_0\,\phi\,T}{M\,\pi}$$

and an angular velocity—

$$\omega = \frac{\pi\,\phi}{T}$$

where—

$$T = \frac{\pi}{\sqrt{\frac{\gamma}{M\,\rho^2}}}$$

We then find—

$$\theta - \theta_0 = \phi \sin \frac{\pi\,t}{T}$$

$$y = \frac{\phi\,Q_0\,T^2}{M\,\pi^2} \sin \frac{\pi\,t}{T}.$$

Thus the path of G is a curve of sines oscillating on both sides of the original level, and with a half amplitude—

$$y_1 = \pm \frac{\phi\,Q_0\,T^2}{M\,\pi^2}.$$

The horizontal length of the oscillations will be—

$$L = T\,v = \pi\,v \sqrt{\frac{M\,\rho^2}{\gamma}}.$$

Angularly the boat will oscillate about the original position θ_0 with a half amplitude equal to $\pm\,\phi$.

It is found that, when the boat is at its lowest level, the axis will have its minimum inclination to the horizon, and *vice versâ,* the consequence of which is, that, when the boat is ascending, the fore-end swings downwards, and when descending it swings upwards.

Let us now imagine lateral resistance to act. The force Q will be subject to fluctuations, and will, apart from the effect of angular motion, be augmented during ascent, and diminished during descent. If the centre of lateral resistance C lies aft of G, the effect will be that the angular motion is always opposed by these fluctuations. This is expressed in the formulæ for resisted motion by the fact that s is negative. Conversely, the fluctuations in Q will assist and augment the angular motion, if C lies forward of G ; and s is, in this case, positive.

The fluctuations in Q will, moreover, cause an oscillatory motion of G, which will combine with that due to angular motion. The two oscillatory motions will mutually react on each other, and the result will be a curve determined by two terms, one containing a sine, the other a cosine.

Resistance Varying as Angular Velocity Included.

The differential equations become—

$$\frac{d^2\theta}{dt^2} + \frac{k}{M\rho^2}\frac{d\theta}{dt} + \frac{\beta}{M\rho^2 v}\frac{dy}{dt} + \frac{\gamma}{M\rho^2}\theta = \frac{\theta_0\gamma - fz}{M\rho^2}$$

$$\frac{d^2y}{dt^2} + \frac{Q_0}{Mv}\frac{dy}{dt} + \frac{Q_0}{M}\theta = \frac{\theta_0 Q_0 + f}{M}$$

The solution of these equations is found to be of the same form as when this resistance was not included, but the exponents are given by—

$$\lambda^3 + \lambda^2\left[\frac{Q_0}{Mv} + \frac{k}{M\rho^2}\right] + \lambda\left[\frac{\gamma}{M\rho^2} + \frac{Q_0 k}{M^2\rho^2 v}\right] + \frac{\epsilon Q_0}{M^2\rho^2 v} = 0.$$

It is seen from this equation that s will be zero for a value of γ which is somewhat smaller than ϵ; in other words, the oscillatory terms may be evanescent, even if the centre of lateral resistance lies a little forward of the centre of gravity.

Stability of Motion.

Suppose the boat deflected a small angle ϕ from the position of equilibrium of steady motion, and left there without any angular velocity. If she now returns to her original position, the motion must be stable.

The solution of the equations of disturbed motion will be of the form—

$$\theta = \theta_0 + A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t} + A_3 e^{\lambda_3 t}$$

$$y = B_0 + B_1 e^{\lambda_1 t} + B_2 e^{\lambda_2 t} + B_3 e^{\lambda_3 t}$$

and the constants are found by the same formula as above, only $L = -\frac{\phi\gamma}{M\rho^2}$ and $K = \phi$.

If, now, β and ϵ are both positive, we know that all the exponential terms will be evanescent, and we get ultimately: $\theta_1 = \theta_0$ and $y_1 = B_0$, *i.e.*, the boat returns to its original inclination, and the motion is stable.

If $\epsilon = 0$, β being still positive, we find $A_1 = 0$ and $B_1 = 0$, and, λ_2 and λ_3 being negative, the motion will still be stable.

If ϵ is negative, and β positive, there will be at least one positive value of λ, and the motion will be ever increasing, and therefore unstable.

If $\beta = 0$, and ϵ positive, we find $A_1 = 0$, $L_3 = 0$, $L_2 = \phi$, and, therefore—

$$\theta = \theta_0 + \phi \cos\frac{\pi}{T} t$$

$$y = B_0 + B_1 e^{-\frac{Q_0}{Mv} t} + M_2 \cos\frac{\pi}{T} t + M_3 \sin\frac{\pi}{T} t,$$

showing that the boat will continue to oscillate indefinitely. The resistance to angular motion will, however, in reality extinguish the oscillations, and we may, therefore, still consider this as a case of stable motion.

If β is negative and numerically $< \epsilon$, which is positive, γ will still be positive, and no positive real value can satisfy the equation for λ, but there will be two imaginary roots having positive real parts, which shows the motion to be oscillatory, ever increasing, and therefore unstable.

This conclusion must, indeed, be somewhat modified, because of the resistance to angular motion, which will permit β to have a small negative value before the motion becomes unstable.

The general conditions of stability of motion are accordingly $\epsilon \overline{\gt} 0$, and $\beta \overline{\gt} 0$, and $\gamma > 0$.

Manœuvring Power.

We have—

$$\frac{d^2\theta}{dt^2} = \frac{S_0 p\, \alpha - \left[(\epsilon + \beta)\,\theta + \frac{\beta}{v}\frac{dy}{dt} + k\frac{d\theta}{dt}\right]}{M\rho^2},$$

or—

$$\frac{d^2\theta}{dt^2} = \frac{\text{Turning moment of rudder} - [\text{stability of motion} + \text{moment of resistance}]}{\text{moment of inertia}}$$

which determines the rate at which angular velocity is acquired or extinguished.

Neither stability of motion nor resistance should be sacrified, in order to attain manœuvring power, which should be secured exclusively through the turning moment of the rudder.

Surplus Buoyancy.

In steady motion, when the conning tower was placed vertically over the centre of gravity, we found—

$$\theta_0 = \frac{wp}{\delta} \quad\text{and}\quad \alpha_0 = \frac{w\gamma}{S_0\,\delta}.$$

Suppose, now, the conning tower to be placed forward, or aft, of G, at a distance from this point. It is supposed, always, that, when the boat is at rest, she will float on even keel at the surface, with only the top of the conning tower projecting above the water. The equations of equilibrium of motion become—

$$S_0 p\, \alpha_0 - \theta_0 \gamma \mp w z = 0,$$
$$S_0 \alpha_0 + w - \theta_0 Q_0 = 0,$$

from which—

$$\frac{\theta_0}{w} = \frac{(p \pm z)}{Q_0(p - a) - \varepsilon}.$$

These equations hold for rudder placed aft. If, now, rudder is placed forward, we get—

$$\frac{\theta_0}{w} = \frac{(p \mp z)}{Q_0(p + a) + \varepsilon}.$$

the upper sign in all equations corresponding to conning tower placed forward, and conversely.

It appears that for obtaining a small value of θ_0, with a great value of w, the conning tower should be placed on same side of centre of gravity as the rudder.

Supposing the conning tower thus placed, it remains to examine whether the forward or the aft position is the best.

The formula is—

$$\frac{\theta_0}{w} = \frac{(p - z)}{Q_0(p \mp a) \mp \varepsilon}$$

upper sign holding for rudder placed aft, lower sign for rudder placed forward. The formula shows that it is most advantageous to place the rudder forward.

In the extreme case when the surplus buoyancy acts directly over the rudder, whether forward or aft, we get $\theta_0 = 0$, *i.e.*, the boat moves on an even keel, and $a_0 = \mp \frac{w}{S_0}$, *i.e.*, turned upwards or downwards in case of an aft or forward rudder respectively.

The Rudders.

Any two rudders placed symmetrically with common turning axis are treated as one rudder.

We have seen that it is unfavourable to surplus buoyancy to place the rudder aft, but the bad effect in this respect may be counteracted by placing the conning tower more aft.

Placing the rudder aft increases the stability of motion, but decreases thereby the manœuvring power; further, the leverage of the rudder is greater aft than forward. The difference in turning moment in the two cases will be—

$$S_0 [\theta (p_1 + p_2) - a (p_2 - p_1)],$$

if p_1 and p_2 are the values of p, when the rudder is placed forward and aft respectively. a being much greater than θ, the gain, if any, cannot be great by changing the rudder from aft to forward, and the manœuvring power will, therefore, not be materially affected.

The rate at which a small variation in the rudder angle influences the angle of inclination of the boat is, for any given position of equilibrium, determined by the value of $\frac{d\theta}{da}$. By differentiating the equation of turning moment, we find—

$$\frac{d\theta}{da} = -\frac{S_0 p}{\gamma}$$

showing the efficacy of the rudder to be directly as the moment of rudder pressure about G, and inversely as the stability of motion.

The effect of the rudder, as regards the limiting condition of the boat, must be the same as that of a disturbing force $S_0 (a - a_0)$, acting at a distance p from G; and the rudder will, consequently, be more effective forward than aft.

When a disturbance of a permanent nature occurs, the instruments will soon indicate that the inclination and the depth of immersion are changed, and the helmsman will, consequently, attempt to bring the boat back to the original depth of immersion, and to find the new position of equilibrium of the boat θ_2, as well as the corresponding rudder angle a_2.

These are given by the formula—

$$S_0 p a_2 - \gamma \theta_2 - f z = 0$$

$$S_0 a_2 - Q_0 \theta + w + f = 0$$

from which—

$$\theta_2 = \frac{f(p+z) + p w}{\delta}$$

$$a_2 = \frac{f z Q_0 + (w+f)\gamma}{S_0 \delta}.$$

It is clear that the task of the helmsman must be more difficult in a submarine boat than in an ordinary ship, where disturbances analogous to those here treated do not occur, and where, consequently, the rudder is always carried at practically the same angle after, as before, a deviation from the right course has taken place.

It is recommended in the first part of this paper, to place the rudder, and consequently also the conning tower, aft (*i.e.*, somewhat aft of G), mainly for practical reasons. We have then (z refers here to the conning tower)—

$$\theta_0 = \frac{S_0 p a_0 + w z}{\gamma} = \frac{w(p-z)}{\delta}$$

and—

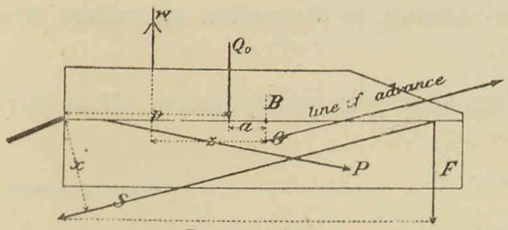
$$a_0 = \frac{w\left[Q_0(a-z) + \epsilon\right]}{S_0 \delta}.$$

This change in the value of θ_0 and a_0 would not, however, affect the form of any of the previous equations.

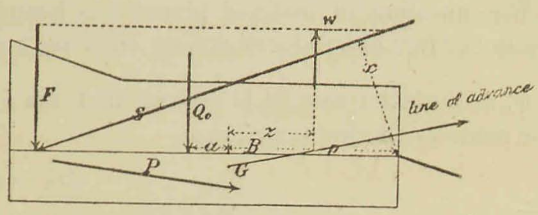
Non-Symmetry.

Imagine a symmetrical boat, altered so as to become unsymmetrical about the horizontal central plane, or the propeller shaft placed away from the axis of the boat, parallel or inclined to the same.

Apart from the system of forces acting on the boat in steady motion, when symmetrical, we shall now find extra forces acting, which are all equivalent to one single force S in the central vertical plane, cutting the axis at some point D. Let the vertical resultant of this force be F, supposed acting downwards.



RUDDER AFT
FIG 8



RUDDER FORWARD.
FIG. 9.

Resolving vertically—

$$S_0 a_0 - F + w - Q_0 \theta_0 = 0.$$

Let x be the distance of S from the rudder stock, and take moments about the same—

$$[Q_0 (p \mp a) \mp \epsilon] \ \theta_0 = w (p \mp z) - S x$$

where upper sign holds for an aft-rudder, lower sign for a fore-rudder.

The first equation shows, that for permitting a great surplus buoyancy with small angle of inclination, F should be acting downwards.

The second equation shows that, for the same reason, the moment of S about the rudder stock should always be of opposite sign to the moment of surplus buoyancy, and should be numerically great.

It is readily seen, from Figs. 8 and 9, that these claims are satisfied thus:—In case of an aft rudder (see Diagram, Fig. 8) the lines of the hull should be rather full in the bottom, and cut away forward in the upper part. The propeller shaft should be placed above the axis, or, preferably, inclined downwards as indicated.

In case of a fore-rudder (see Diagram, Fig. 9) the dispositions should be the converse, except that the propeller shaft should also here be inclined downwards if inclined at all.

It is unavoidable that S should change both in magnitude and direction with the speed, and probably also with the depth of immersion, and it is probable, therefore, that it will cause disturbances in steering. Non-symmetry should therefore be avoided, at any rate in experimental boats.

Heeling.

If the boat is heeled a small angle ψ, and the vertical rudder is laid at an angle a, then if $S_0 a$ is the pressure on the rudder when the boat is upright, the pressure will now have a vertical component $S_0 a \psi$ approximately. This force will be equivalent to the action of a horizontal rudder of same area as the vertical rudder, and laid at an angle $a \psi$.

On Even Keel.

If surplus buoyancy is to be preserved on even keel, and, if not acting just above the rudder, special means must be provided to produce a permanent downhaul force. This may be created either by inclined planes or rudders, or by vertical propellers.

The question of inclined planes has been discussed already in the section on rudders, where midship rudders and rudders at both ends are mentioned.

If, in the latter case, it is desired that the four rudders should, in steady motion, all be carried at the same angle, we must have—

$$(S_{01} + S_{02})\, a_0 = w$$

and—

$$S_{01}\, a_0\, p_1 = S_{02}\, a_0\, p_2$$

i.e., the areas of the rudders must be inversely as their distances from the centre of gravity.

The equations for disturbed motion and their solution will be the same as if the boat were inclined, only $\theta_0 = 0$.

Downhaul propellers entail a great expenditure of the very limited store of energy which the boat possesses for submarine propulsion, and they take up much valuable space. They must be generally used in conjunction with a horizontal rudder, by means of which the boat is kept strictly horizontal.

Rising or sinking is effected by varying the speed of the propellers.

The service must in such case be complicated, because both rudder and propellers have to be manœuvred at the same time.

If surplus buoyancy is not to be preserved, the vertical movements may be effected by means of a pump, which produces variations in displacement. The action of such pump is not so quickly reversed as in case of rudders or propellers.

This method has been used by many earlier inventors for keeping the depth under way, and, as it appears, never with success, at any rate not in large boats (*Le Plongeur*). It presents the advantage which will probably make such a pump indispensable in any boat not having downhaul propellers, that it can be used when the boat is at rest (*Le Goubet*).

By all three methods here mentioned the boat must, when rising or sinking, be moved in the direction of her greatest resistance; while by inclining the boat, it can always be moved in direction of its least resistance or nearly so.

Numerical Examples.

The following data are taken from a design worked out by the author.
Given—

$W = 100$ tons. $v = 6$ knots $= 10$ ft. per second.
Area of rudder $= 20$ sq. ft. $p = 34$ ft.
Area of central horizontal section $= 700$ sq. ft.
$B\,G = \cdot 75$ ft. $a = 4\cdot 88$ ft. $w = \cdot 5$ tons. $\rho = 20$ ft.

Assume the lateral resistance given by the formula $Q_0 = \Sigma\,(k\,A\,v^2)$, and assume k in case of the hull to be $= 1$, and in case of the rudder to be $= 2\,25$, then—

$$Q_0 = \frac{1 \cdot 00 \times 700 \times \overline{10}^2 + 2 \cdot 25 \times 20 \times \overline{10}^2}{2240} = 33 \text{ tons}$$

and—

$$S_0 = \frac{2 \cdot 25 \times 20 \times 10^2}{2240} = 2 \text{ tons.}$$

Moreover we find—

$\epsilon = 75$ ft. tons. $\beta = 161$ ft. tons, and $\gamma = 236$ ft. tons. $M = 3 \cdot 125$. $M v = 31 \cdot 25$. $M \rho^2 = 1250$. $\delta = 886$. $\theta_0 = \cdot 0192$, about 1°. $\alpha_0 = \cdot 0666$, about 4°. $F = 1394$. $F = 18 \cdot 6\, \epsilon$,

wherefore the motion will be oscillatory for all values of γ.

(1) *A downwards Force acting forward.*

Given—

$f = - \cdot 075$ tons. $\quad z = + 30$ ft. $\quad f z = 2 \cdot 25$ ft. tons,

the equation for the exponents is—

$$\lambda^3 + \lambda^2\, 1 \cdot 056 + \lambda \cdot 1888 + \cdot 0634 = 0,$$

which gives—

$$A = - \cdot 18290. \qquad B = + \cdot 08413. \qquad \frac{B^2}{4} + \frac{A^3}{27} = + \cdot 001542.$$

$$\left.\begin{matrix} p_1^3 \\ q_1^3 \end{matrix}\right\} = - \cdot 04207 \pm \sqrt{\cdot 001542} = \left\{\begin{matrix} - \cdot 00280 \\ - \cdot 08134 \end{matrix}\right.$$

$$p_1 = - \cdot 1408. \qquad q_1 = - \cdot 4333.$$

$$\lambda_1 = - \cdot 1408 - \cdot 4333 - \cdot 3520 = - \cdot 9261$$

$$\left.\begin{matrix} p_2 \\ p_3 \end{matrix}\right\} = \cdot 0704 \mp 1220 i. \qquad \left.\begin{matrix} q_2 \\ q_3 \end{matrix}\right\} = \cdot 2166 \mp \cdot 3752 i.$$

$$\lambda_2 = - \cdot 0649 + \cdot 2532 i. \qquad s = - \cdot 0649$$

$$\lambda_3 = - \cdot 0649 - \cdot 2532 i. \qquad u = + \cdot 2532.$$

$$A_0 = + \cdot 0192 + \cdot 0349 = + \cdot 0541. \qquad L = + \cdot 0018. \qquad K = - \cdot 0349.$$

$$A_1 = - \cdot 00072. \qquad L_2 = - \cdot 0356. \qquad L_3 = - \cdot 0118.$$

$$B_1 = - \cdot 0632.$$

$$\frac{A_1}{\lambda_1} + \frac{A_2}{\lambda_2} + \frac{A_3}{\lambda_3} = + \cdot 0812. \qquad B_0 = + 1 \cdot 1898.$$

$$M_2 = - 1 \cdot 1267. \qquad M_3 = + \cdot 9472.$$

$$+ \left[\frac{f}{Q_0} + \frac{f(a+z)}{\epsilon} \right] v = - \cdot 3715.$$

The expression for θ and y are:

$$\theta = - \cdot 00072\, e^{-\cdot 9261 t} - e^{-\cdot 0649 t} [\cdot 0356 \cos (\cdot 2532\, t) + \cdot 0118 \sin (\cdot 2532\, t)] + \cdot 0541.$$

$$y = - \cdot 0632\, e^{-\cdot 9261 t} - e^{-\cdot 0649 t} [1 \cdot 1267 \cos (\cdot 2532\, t) - \cdot 9472 \sin (\cdot 2532\, t)] - \cdot 3715\, t + 1 \cdot 1898.$$

The limiting conditions are—

$$\theta_1 = + \cdot 0541, \text{ about } 3^\circ \text{ inclination to the horizon downwards.}$$

$$\left(\frac{dy}{dt}\right)_1 = \cdot 3715 \text{ ft. per second downwards, or about 22 ft. per minute.}$$

The steepness of the trajectory is $\frac{1}{v}\left(\frac{dy}{dt}\right)_1 = \cdot 03715$, about 2° inclined to the horizon. The angle between the axis and the trajectory is $\zeta = \cdot 0541 - \cdot 0372 = \cdot 0169$, about 1°, which is the drift angle in the limiting disturbed motion. ζ differs from the drift angle in steady motion, θ_0, only by the small amount: $\cdot 0192 - \cdot 0169 = \cdot 0023 = \frac{f}{Q_0}$, about 8′. The resistance, and consequently the speed along the trajectory, must therefore be practically the same in both cases.

Before coming to rest, the boat will oscillate with a period—

$$T = \frac{\pi}{\cdot 2532} = 12\cdot 4 \text{ seconds.}$$

The curve 1 on Plate shows the path of the centre of gravity in this case. The ordinates are on a larger scale than the abcissæ in order to show the undulations more clearly.

(2) *A downward Force acting Aft.*

Let $z = -30$ ft., as before $f = - \cdot 075$ tons.

The values of λ_1, s, and u remain unaltered.

$$\frac{f(a+z)}{\varepsilon} = - \cdot 0251. \qquad A_0 = - \cdot 0059.$$

$$L = - \cdot 0018. \qquad K = \cdot 0251. \qquad A_1 = - \cdot 00010.$$

$$L_2 = + \cdot 0252. \qquad L_3 = + \cdot 0061. \qquad B_1 = - \cdot 0088.$$

$$\frac{A_1}{\lambda_1} + \frac{A_2}{\lambda_2} + \frac{A_3}{\lambda_3} = - \cdot 0464. \qquad B_0 = - \cdot 6804.$$

$$M_2 = + \cdot 6892. \qquad M_3 = - \cdot 7577. \qquad \left[\frac{f}{Q_0} + \frac{f(a+z)}{\varepsilon}\right] v = + \cdot 2285.$$

The expressions for θ and y become—

$$\theta = - \cdot 0001\, e^{-\cdot 0261 t} + e^{-\cdot 0649 t}\left[\cdot 0252 \cos(\cdot 2532\, t) + \cdot 0061 \sin(\cdot 2532\, t)\right] - \cdot 0059.$$

$$y = - \cdot 0088\, e^{-\cdot 0261 t} + e^{-\cdot 0649 t}\left[\cdot 6892 \cos(\cdot 2532\, t) - \cdot 7577 \sin(\cdot 2532\, t)\right] + \cdot 2285\, t - \cdot 6804.$$

The limiting conditions are—

$$\theta_1 = - \cdot 0059, \text{ about } \tfrac{1}{3}^\circ \text{ upwards.}$$

$$\left(\frac{dy}{dt}\right)_1 = \cdot 2285 \text{ ft. per second upwards, or about 13 ft. per minute.}$$

The steepness of the trajectory is in the limit smaller than when the weight was acting forward being now only—

$$\frac{1}{v}\left(\frac{dy}{dt}\right)_1 = \cdot 02285, \text{ about } 1\tfrac{1}{2}^\circ \text{ inclined to the horizon.}$$

The period of oscillations will be the same.

It will be noticed on Plate , curve 2, that the path of the centre of gravity falls at first a little below the horizontal line, and then it crosses this line, ascends, and ends in a straight line after some undulations.

The following Table shows the limiting conditions in various cases :—

Disturbance.	Angular Deflection.	Change of Depth of Immersion in One Minute.
·075 tons added 30 ft. forward of G	2° downwards.	22 ft. downwards.
·075 tons added 30 ft. aft of G	$1\frac{1}{3}$° upwards.	13 ft. upwards.
·075 tons added at G	$\frac{1}{2}$° downwards.	4 ft. downwards.
·075 tons added at centre of lateral resistance ...	0	1 ft. downwards.
·075 tons moved through 60 ft. from aft to forward	$3\frac{1}{2}$° downwards.	36 ft. downwards.

To Illustrate Kaptain G. W. Hovgaard's Paper: The Motion of Submarine Boats in the Vertical Plane.

Curves showing Path of Centre of Gravity of a Submarine Boat, which, being in Steady Motion along a Horizontal Line, is Disturbed by:

1. A weight added forward
2. A weight added aft: